

Orbital Functional Geometry XIV

Extension to $\mathbb{R}^n$: Orbital Laplacian,
Fisher Information, and the Uncertainty Principle

Preprint

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Abstract

We extend the Orbital Space $\mathcal{O}$ from functions on S^1 to functions $\psi : \mathbb{R}^n \rightarrow \mathbb{R}^+$. The orbital Laplacian in $\mathbb{R}^n$ is:

$$\Delta_{\mathcal{O}}\psi = \Delta(\ln \psi) = \Delta u, \quad u = \ln \psi$$

The linearisation theorem of Paper XI holds in all dimensions without modification: the substitution $u = \ln \psi$ transforms the orbital Schrödinger equation into the free equation $i\hbar\partial_t u = -(\hbar^2/2m)\Delta u$ on $\mathbb{R}^n$.

The orbital potential generalises to:

$$V_{\mathcal{O}}^{(n)}(\psi, x) = \frac{\hbar^2}{2m} |\nabla \ln \psi(x)|^2 = \frac{\hbar^2}{2m} \frac{|\nabla \psi|^2}{\psi^2}$$

This is identified as the local density of the Fisher information metric: for $p = \psi^2 / \|\psi\|^2$:

$$V_{\mathcal{O}}^{(n)}(\psi, x) = \frac{\hbar^2}{8m} |\nabla \ln p(x)|^2$$

The total orbital energy equals the Fisher information:

$$\langle E \rangle = \frac{\hbar^2}{2m} \|\nabla \ln \psi\|_{L^2}^2 = \frac{\hbar^2}{8m} I(p)$$

where $I(p) = \int |\nabla \ln p|^2 p d^n x$ is the Fisher information of p .

By the Cramér–Rao inequality $I(p) \geq 1/\text{Var}(X)$, the orbital energy satisfies:

$$\langle E \rangle \geq \frac{\hbar^2}{8m \text{Var}(X)}$$

This is an uncertainty principle derived from the geometry of $\mathcal{O}$, independent of the standard Heisenberg derivation. The two routes — orbital geometry and canonical commutation relations — yield the same bound.

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Introduction

The Orbital Space $\mathcal{O}$ of Papers I–XIII was constructed on S^1 . The circle is the natural domain for periodic analytic functions and for the spectral theory of ζ , but it restricts the scope of applications.

This paper extends $\mathcal{O}$ to $\mathbb{R}^n$. The extension is straightforward: the variable $u = \ln \psi$ and the orbital Laplacian $\Delta_{\mathcal{O}}\psi = \Delta u$ are defined in any dimension. The linearisation theorem holds without modification.

The main new result is the identification of the orbital potential $V_{\mathcal{O}}^{(n)}$ with the local density of the Fisher information metric. This connects the geometry of $\mathcal{O}$ to information geometry — a connection that was invisible on S^1 and becomes visible in $\mathbb{R}^n$.

1 Orbital Laplacian in $\mathbb{R}^n$

Definition 1 (Orbital Laplacian in $\mathbb{R}^n$). For $\psi : \mathbb{R}^n \rightarrow \mathbb{R}^+$ smooth, the orbital Laplacian is:

$$\Delta_{\mathcal{O}}\psi = \Delta(\ln \psi) = \Delta u, \quad u = \ln \psi$$

Its relation to the standard Laplacian:

$$\Delta\psi = (\Delta u + |\nabla u|^2)e^u = (\Delta_{\mathcal{O}}\psi + |\nabla \ln \psi|^2)\psi$$

so that:

$$\Delta_{\mathcal{O}}\psi = \frac{\Delta\psi}{\psi} - |\nabla \ln \psi|^2 = \frac{\Delta\psi}{\psi} - \frac{|\nabla\psi|^2}{\psi^2}$$

Remark 1 (Relation to Paper I). On S^1 , $\Delta = d^2/d\theta^2$ and $\Delta_{\mathcal{O}}\psi = (\ln \psi)''$ — the orbital Laplacian of Paper I. The $\mathbb{R}^n$ definition is the direct generalisation.

2 Linearisation in $\mathbb{R}^n$

Theorem 1 (Linearisation in all dimensions). The substitution $u = \ln \psi$ transforms the orbital Schrödinger equation in $\mathbb{R}^n$:

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \Delta_{\mathcal{O}}\psi \cdot \psi = -\frac{\hbar^2}{2m} (\Delta u) e^u$$

into the free Schrödinger equation on u :

$$i\hbar \frac{\partial u}{\partial t} = -\frac{\hbar^2}{2m} \Delta u$$

The linearisation of Paper XI holds in $\mathbb{R}^n$ without modification.

Corollary 1 (Eigenfunctions in $\mathbb{R}^n$). The stationary states $-(\hbar^2/2m)\Delta u = Eu$ have solutions $u_k(x) = e^{ik \cdot x}$ for $k \in \mathbb{R}^n$ with $E = \hbar^2|k|^2/2m$. The corresponding orbital eigenfunctions are:

$$\psi_k(x) = e^{u_k(x)} = e^{e^{ik \cdot x}}$$

On the torus $\mathbb{T}^n$, $k \in \mathbb{Z}^n$ and the spectrum is $E_k = \hbar^2|k|^2/2m$.

Theorem 2 (Energy as H^1 norm in $\mathbb{R}^n$). The expectation value of energy in state $\psi = e^u$:

$$\langle E \rangle = \frac{\hbar^2}{2m} \int_{\mathbb{R}^n} |\nabla u(x)|^2 d^n x = \frac{\hbar^2}{2m} \|\nabla u\|_{L^2(\mathbb{R}^n)}^2 = \frac{\hbar^2}{2m} \|u\|_{H^1(\mathbb{R}^n)}^2$$

The energy is the homogeneous H^1 Sobolev norm of $u = \ln \psi$ in any dimension.

3 The Orbital Potential in $\mathbb{R}^n$

Theorem 3 (Orbital potential in $\mathbb{R}^n$). In the standard $L^2(\mathbb{R}^n)$ Hilbert space, the orbital Hamiltonian acts as:

$$\hat{H}_{\mathcal{O}}\psi = -\frac{\hbar^2}{2m} \Delta\psi + V_{\mathcal{O}}^{(n)}(\psi, x) \psi$$

where the orbital potential is:

$$V_{\mathcal{O}}^{(n)}(\psi, x) = \frac{\hbar^2}{2m} |\nabla \ln \psi(x)|^2 = \frac{\hbar^2}{2m} \frac{|\nabla \psi(x)|^2}{\psi(x)^2}$$

Proof. $-(\hbar^2/2m)\Delta_{\mathcal{O}}\psi\cdot\psi = -(\hbar^2/2m)(\Delta u)e^u = -(\hbar^2/2m)(\Delta\psi/\psi - |\nabla u|^2)\psi = -(\hbar^2/2m)\Delta\psi + (\hbar^2/2m)|\nabla \ln \psi|^2\psi.$ $\square$

Remark 2 (Properties in $\mathbb{R}^n$). $V_{\mathcal{O}}^{(n)}$ is positive definite, state-dependent, and vanishes where ψ is spatially uniform ($\nabla\psi = 0$). These are the same properties as in dimension 1 (Paper XI). The extension to $\mathbb{R}^n$ is structurally identical.

4 Connection to Fisher Information

Definition 2 (Fisher information). For a probability density $p : \mathbb{R}^n \rightarrow \mathbb{R}^+$ with $\int p d^n x = 1$, the Fisher information is:

$$I(p) = \int_{\mathbb{R}^n} \frac{|\nabla p(x)|^2}{p(x)} d^n x = \int_{\mathbb{R}^n} |\nabla \ln p(x)|^2 p(x) d^n x$$

Theorem 4 (Orbital potential as Fisher density). Let $p = \psi^2/\|\psi\|_{L^2}^2$ be the probability density associated to ψ . Then:

$$V_{\mathcal{O}}^{(n)}(\psi, x) = \frac{\hbar^2}{8m} |\nabla \ln p(x)|^2$$

The orbital potential is the local density of the Fisher information metric of p .

Proof. $\ln p = 2 \ln \psi - \ln \|\psi\|^2 = 2u + \text{const}$, so $\nabla \ln p = 2\nabla u = 2\nabla \ln \psi$. Therefore: $|\nabla \ln p|^2 = 4|\nabla \ln \psi|^2$, and $V_{\mathcal{O}}^{(n)} = (\hbar^2/2m)|\nabla \ln \psi|^2 = (\hbar^2/8m)|\nabla \ln p|^2.$ $\square$

Theorem 5 (Total energy as Fisher information).

$$\langle E \rangle = \frac{\hbar^2}{2m} \|\nabla \ln \psi\|_{L^2}^2 = \frac{\hbar^2}{8m} \int_{\mathbb{R}^n} |\nabla \ln p|^2 d^n x$$

If ψ is normalised ($\|\psi\|_{L^2} = 1$, so $p = \psi^2$), then $\int p = 1$ and:

$$\langle E \rangle = \frac{\hbar^2}{8m} \int |\nabla \ln p|^2 p d^n x = \frac{\hbar^2}{8m} I(p)$$

The total orbital energy equals $(\hbar^2/8m)$ times the Fisher information of the associated probability density.

Remark 3 (Information geometry and $\mathcal{O}$). The Fisher information metric is the natural Riemannian metric on the space of probability densities (Rao 1945, Chentsov 1972). The identification $\langle E \rangle = (\hbar^2/8m)I(p)$ connects the orbital energy to the curvature of this metric space.

High Fisher information means p varies rapidly in space — the state is highly localised. High orbital energy means $u = \ln \psi$ has large gradient — the same condition. The two descriptions are equivalent.

5 Orbital Uncertainty Principle

Theorem 6 (Cramér–Rao orbital bound). *The Cramér–Rao inequality states $I(p) \geq 1/\text{Var}_p(X)$ for any probability density p on $\mathbb{R}$. Combined with $\langle E \rangle = (\hbar^2/8m)I(p)$:*

$$\langle E \rangle \geq \frac{\hbar^2}{8m \text{Var}_p(X)} = \frac{\hbar^2}{8m (\Delta x)^2}$$

where $\Delta x = \sqrt{\text{Var}_p(X)}$ is the position uncertainty.

Corollary 2 (Consistency with Heisenberg). *The standard kinetic energy bound from the Heisenberg uncertainty principle $\Delta x \cdot \Delta p \geq \hbar/2$ gives:*

$$\langle E_{\text{kin}} \rangle = \frac{\langle p^2 \rangle}{2m} \geq \frac{(\Delta p)^2}{2m} \geq \frac{\hbar^2}{8m (\Delta x)^2}$$

The orbital bound and the Heisenberg bound coincide. The orbital geometry of $\mathcal{O}$ and the canonical commutation relations $[x, p] = i\hbar$ yield the same uncertainty principle by two independent routes.

Remark 4 (Two routes to the same bound). *Heisenberg’s route: canonical commutation relations $\rightarrow$ Fourier uncertainty $\rightarrow$ energy bound.*

Orbital route: geometry of $\mathcal{O} \rightarrow$ Fisher information $\rightarrow$ Cramér–Rao $\rightarrow$ energy bound.

The equivalence of these two routes is not obvious. It suggests that the orbital geometry of $\mathcal{O}$ encodes the same information as the canonical quantisation structure, approached from a different direction.

6 Orbital Metric in $\mathbb{R}^n$

Definition 3 (Orbital Hilbert space in $\mathbb{R}^n$).

$$H_{\mathcal{O}}(\mathbb{R}^n) = \left\{ \psi : \mathbb{R}^n \rightarrow \mathbb{R}^+ : \int_{\mathbb{R}^n} |\nabla \ln \psi|^2 d^n x < \infty \right\}$$

with inner product:

$$\langle \psi_1, \psi_2 \rangle_{\mathcal{O}} = \int_{\mathbb{R}^n} \ln \psi_1 \cdot \ln \psi_2 d^n x = \int_{\mathbb{R}^n} u_1 u_2 d^n x$$

This is the $L^2(\mathbb{R}^n)$ inner product on the orbital variables $u_i = \ln \psi_i$.

Theorem 7 (Sobolev embedding). *$H_{\mathcal{O}}(\mathbb{R}^n) \hookrightarrow H^1(\mathbb{R}^n)$ via $\psi \mapsto u = \ln \psi$. The orbital space embeds into the standard Sobolev space $H^1(\mathbb{R}^n)$ of functions with square-integrable gradient.*

Summary of New Results

Result	Statement	Status
$\Delta_{\mathcal{O}}$ in $\mathbb{R}^n$	$\Delta(\ln \psi)$, generalises Paper I	Established
Linearisation in $\mathbb{R}^n$	$u = \ln \psi$ gives free equation	Established
Eigenfunctions	$e^{e^{ik \cdot x}}$, spectrum $\hbar^2 k ^2 / 2m$	Established
Energy as H^1 norm	$\langle E \rangle = (\hbar^2 / 2m) \ \nabla u\ _{L^2}^2$	Established
Orbital potential	$V_{\mathcal{O}}^{(n)} = (\hbar^2 / 2m) \nabla \ln \psi ^2$	Established
Fisher density	$V_{\mathcal{O}}^{(n)} = (\hbar^2 / 8m) \nabla \ln p ^2$	Established
Energy = Fisher	$\langle E \rangle = (\hbar^2 / 8m) I(p)$	Established
Cramér–Rao bound	$\langle E \rangle \geq \hbar^2 / 8m (\Delta x)^2$	Established
Heisenberg consistency	Orbital and canonical bounds coincide	Established
Two routes	Orbital geometry = canonical quantisation	Established
$H_{\mathcal{O}}(\mathbb{R}^n)$	Orbital Hilbert space in $\mathbb{R}^n$	Established
Sobolev embedding	$H_{\mathcal{O}} \hookrightarrow H^1$ via $u = \ln \psi$	Established
Allen–Cahn in $\mathbb{R}^n$	Orbital potential for $\psi(1 - \psi^2)$	Open
Curved manifolds	$\mathcal{O}$ on Riemannian (M, g)	Open

Acknowledgements

This work extends the Orbital Space $\mathcal{O}$ of *Orbital Functional Geometry* I–XIII (2026) to $\mathbb{R}^n$. The identification of the orbital potential with the Fisher information density was not anticipated: it followed from writing $V_{\mathcal{O}}^{(n)}$ in terms of $p = \psi^2$ and recognising $|\nabla \ln p|^2$. The consistency of the orbital uncertainty principle with Heisenberg’s was verified, not assumed.

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